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The Ashkin–Teller quantum chain and conformal invariance

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Abstract. We study the finite-size effects for the four-state one-dimensional quantum chain introduced by Kohmoto *et al* using different boundary conditions. From the corrections to the ground-state energy for periodic boundary conditions we verify that the central charge of the Virasoro algebra is equal to one in the whole domain of criticality. Our numerical study suggests a new relation between the corrections to the ground-state energy with free boundary conditions and with periodic boundary conditions. Various critical exponents including the surface ones and those corresponding to the para-fermions are determined.

1. Introduction

In this paper we study finite-size effects on the spectrum of the four-state one-dimensional quantum chain:

$$H=-\frac{1}{2(1+\varepsilon)}\left(\sum_{i=1}^{N}[\sigma_i+(\sigma_i)^3+\varepsilon(\sigma_i)^2]+\lambda\sum_{i=1}^{N}[\Gamma_i\Gamma_{i+1}^{+}+\Gamma_i^{+}\Gamma_{i+1}+\varepsilon(\Gamma_i)^2(\Gamma_{i+1})^2]\right) \tag{1.1}$$

where

$$\sigma=\begin{pmatrix}1&0&0&0\\0&\mathrm{i}&0&0\\0&0&-1&0\\0&0&0&-\mathrm{i}\end{pmatrix}\qquad \Gamma=\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}. \tag{1.2}$$

The self-dual Hamiltonian (1.1) was proposed by Kohmoto *et al* (1981) and represents one of the possible Hamiltonian versions of the Ashkin-Teller (1943) model. In equation (1.1) λ plays the role of the inverse of the temperature T, ε is a coupling constant and N denotes the number of sites.

The phase diagram of this system is shown in figure 1. It consists of a fully ordered ferromagnetic region (labelled I), a partially ordered phase (II) separated by two Ising lines from the ferromagnetic phase (I) and the paramagnetic phase (III), an antiferromagnetic phase (IV) and a critical fan region (V). The border between the critical fan and regions I and III is of Kosterlitz-Thouless type. If we stay on the self-dual line $\lambda=1$ and vary ε, we move from the four-state Potts model ($\varepsilon=1$) to the Ising model ($\varepsilon=0$), the Kosterlitz-Thouless model ($\varepsilon=-\sqrt{2}/2$) and a first-order phase transition ($\varepsilon=-1$). It was argued by Kohmoto *et al* (1981) (see also Kadanoff and Brown 1979) that on the $\lambda=1$ line the scaling dimensions of the energy operator (x_t), the magnetic field operator (x_H) and the electric field operator (x_P) are

$$x_t=x_T\qquad x_H=\tfrac{1}{8}\qquad x_P=\tfrac{1}{4}x_T \tag{1.3}$$

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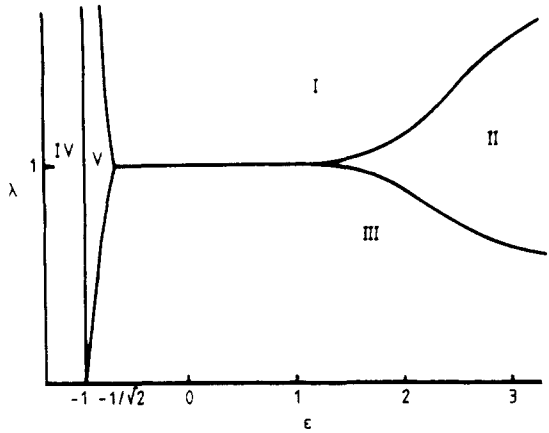

Figure 1. Phase diagram of the Ashkin-Teller quantum chain. I is the ferromagnetic region, II a partially ordered phase, III the paramagnetic phase, IV the antiferromagnetic phase and V the critical fan.

where the function $x_T = x_T(\varepsilon)$ is

$$x_T = \pi/2 \cos^{-1}(-\varepsilon). \tag{1.4}$$

In the region $-\sqrt{2}/2 \leqslant \varepsilon \leqslant 1$, $\lambda = 1$ predictions (1.3) were shown to be correct by high-temperature expansions (Kohmoto *et al* 1981) and by finite-size scaling (Iglói and Sólyom 1984, Alcaraz and Drugowich de Felicio 1984). The same authors have also computed several critical points on the curves separating regions V and I, and I and II respectively (these curves are not known analytically).

Now, it has been shown recently (von Gehlen *et al* 1986) that at second-order phase transition points of a one-dimensional quantum chain the powerful consequences of conformal invariance can be exploited. Conformal invariance for a one-dimensional chain with the linear coordinate x arises in the following way: the Hamiltonian defines an evolution in the Euclidean time coordinate τ. Scaling H by an appropriate chosen factor ξ, one obtains conformal invariance in the space of x and τ.

The aim of this paper is to further explore the model given by equation (1.1) using conformal invariance and finite-size scaling for different boundary conditions. In this way one can identify more operators and find their scaling dimensions. We can thus determine the surface exponents and the exponents corresponding to the para-fermionic operators (Fradkin and Kadanoff 1980). At the same time we have also studied the behaviour of the critical exponents inside the critical fan ($-1 < \varepsilon \leqslant -\sqrt{2}/2$) and we have found a few surprises.

It was recently pointed out by Cardy (1986a) and Blöte *et al* (1986) that for periodic boundary conditions at the critical point, the finite-size corrections to the ground-state energy are related to the central charge of the Virasoro algebra corresponding to the given phase transition. We have studied these corrections and have determined the central charge not only for the $\lambda = 1$ ($-\sqrt{2}/2 \leqslant \varepsilon \leqslant 1$) line but also for the critical fan region. We have repeated this study taking free boundary conditions and have discovered empirically that also in the free boundary case the finite-size corrections are similarly related to the central charge of the Virasoro algebra, the difference being just a factor 4.

Last but not least, our interest in this model comes from the fact that recently we studied six- and eight-state models generalising the Hamiltonian (1.1) (von Gehlen and Rittenberg 1986a). There we have also found critical exponents continuously varying with the coupling constant and we were interested in comparing the behaviour of the critical exponents.

This paper is organised as follows. In § 2 we summarise the implications of conformal invariance on the finite-size scaling behaviour of various energy gaps (Cardy 1984a, 1986b, von Gehlen *et al* 1986) and we show how to obtain the critical exponents. We also discuss the corrections to the ground-state energy. In § 3 we present our numerical results as well as several conjectures suggested by the data. Our conclusions are given in § 4.

2. Conformal invariance and finite one-dimensional chains

Before proceeding with a summary of the general theory let us further specify the Hamiltonian (1.1) through boundary conditions. If we take

$$\Gamma_{N+1} = \exp(\mathrm{i}\pi\tilde{Q}/2)\Gamma_1 \qquad (\tilde{Q} = 0, 1, 2, 3) \tag{2.1}$$

we call the corresponding Hamiltonian $H^{(\tilde{Q})}$. If $\Gamma_{N+1} = 0$ (free boundary conditions) we call the Hamiltonian $H^{(\mathrm{F})}$. The Hamiltonians commute with the Z_4 charge operator:

$$\hat{Q} = \sum_{i=1}^{N} q_i \pmod 4 \tag{2.2}$$

where

$$q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \tag{2.3}$$

(we follow closely the notations of von Gehlen *et al* (1986)).

Because of charge conservation, the Hamiltonians $H^{(\tilde{Q})}$ and $H^{(\mathrm{F})}$ split into four charge sectors and we denote the corresponding matrices by $H_Q^{(\tilde{Q})}$ and $H_Q^{(\mathrm{F})}$ ($Q = 0, 1, 2$ and 3). Self-duality and the dihedral symmetry of the problem give the following relations between these matrices:

$$H_Q^{(\tilde{Q})} = H_{\tilde{Q}}^{(Q)} \qquad H_1^{(\tilde{Q})} = H_3^{(\tilde{Q})} \qquad (Q, \tilde{Q} = 0, 1, 2, 3)$$
$$H_1^{(\mathrm{F})} = H_3^{(\mathrm{F})}. \tag{2.4}$$

The eigenvalues of the matrices $H_Q^{(\tilde{Q})}$ will be denoted by $E_Q^{(\tilde{Q})}(P, r)$ ($r = 0, 1, 2, \ldots$) where $E_Q^{(\tilde{Q})}(P, 0) < E_Q^{(\tilde{Q})}(P, 1)$, etc. Here P denotes the momentum of the state†.

† For $\tilde{Q} \neq 0$ on N sites and in the charge sector Q, eigenstates of H with momentum P may be defined by

$$N^{1/2}|P, Q, \ldots\rangle = |\alpha_1, \alpha_2, \ldots, \alpha_N\rangle + \exp(2\pi\mathrm{i}c_1/nN)|\alpha_2, \alpha_3, \ldots, \alpha_N, \alpha_1\rangle + \ldots$$
$$+ \exp(2\pi\mathrm{i}c_{N-1}/nN)|\alpha_N, \alpha_1, \ldots, \alpha_{N-1}\rangle$$

with

$$c_k = \tilde{Q}N(\alpha_1 + \ldots + \alpha_k) - (Q\tilde{Q} + Pn)k$$

if the symmetry is Z_n and the states are labelled by $\alpha_1, \alpha_2, \ldots, \alpha_N \in Z_n$.

Similarly, $E_Q^{(F)}(r)$ denotes the spectra of the matrices $H_Q^{(F)}$. For brevity, the ground-state energies of the Hamiltonians with periodic ($\tilde{Q}=0$) and free boundary conditions will be denoted by

$$E_0^{(P)} = E_0^{(0)}(0,0) \qquad E_0^{(F)} = E_0^{(F)}(0). \tag{2.5}$$

In a conformal invariant theory, at the critical point the various energy gaps can be used to determine the critical exponents of various operators (Cardy 1986b). We denote by x the scaling dimension and by s the spin of an operator. We can determine the scaling dimensions of the two order operators of our system using the relations

$$P_Q^{(0)} = \lim_{N\to\infty} N(E_Q^{(0)}(0,0) - E_0^{(P)}) = 2\pi\xi x_Q \qquad (Q=1,2) \tag{2.6}$$

(because of the symmetry relation (2.4), $x_1 = x_3$). As we will see in the next section, x_1 will be identified with x_H and x_2 with x_P of equation (1.3). The order operators are spinless. In equation (2.6) ξ represents an unknown constant which appears because one can always multiply the Hamiltonian (1.1) by an arbitrary factor.

The surface exponents of the order operators can be determined using the gaps

$$P_Q^{(F)} = \lim_{N\to\infty} N(E_Q^{(F)}(0) - E_0^{(F)}) = \pi\xi x_{Q,s} \qquad (Q=1,2). \tag{2.7}$$

The scaling dimension of the energy density operator x_t is obtained from the equation

$$R^{(0)} = \lim_{N\to\infty} N(E_0^{(0)}(0,1) - E_0^{(P)}) = 2\pi\xi x_t. \tag{2.8}$$

The normalisation factor ξ can be determined from

$$R^{(Q)} = \lim_{N\to\infty} N(E_Q^{(0)}(1,0) - E_Q^{(0)}(0,0)) = 2\pi\xi \qquad (Q=1,2). \tag{2.9}$$

Let us stress that in this paper we have not done a complete analysis of higher excitations for various momenta, as was done for the three-state model (von Gehlen and Rittenberg 1986b). Such an analysis would have revealed many more operators. We have confined ourselves only to the lowest excitations and thus to operators with the smallest scaling dimensions.

We now turn to the para-fermionic operators (Fradkin and Kadanoff 1980) with spin $s = \tilde{Q}/4$ and scaling dimensions $x_{Q,\tilde{Q}}$. The values of $x_{Q,\tilde{Q}}$ can be obtained from the equations

$$P_Q^{(\tilde{Q})} = \lim_{n\to\infty} N(E_Q^{(\tilde{Q})}(0,0) - E_0^{(P)}) = 2\pi\xi x_{Q,\tilde{Q}}. \tag{2.10}$$

As is by now well known, the possible values of the scaling dimensions and spins are limited in two dimensions (or equivalently for a one-dimensional quantum chain) to a few discrete values. This is the case if the phase transition corresponds to a Virasoro algebra with a central charge $c<1$ (Belavin *et al* 1984, Friedan *et al* 1984). Since for our Hamiltonian (1.1) we know already that some exponents vary continuously (see x_t and x_p in equation (1.3)) it results that $c \geqslant 1$. Actually one knows more. From the known equivalence between the Ashkin-Teller model and the Thirring (1958) model (see Drugowich de Felicio and Köberle 1982) for which the central charge can be computed analytically (Fubini *et al* 1973), we learn that $c=1$. It was suggested by

Blöte *et al* (1986) (see also Cardy 1986a) that the finite-size corrections to the ground-state energy $E_0^{(P)}$ (periodic boundary conditions) are related to the central charge c:

$$-E_0^{(P)}/N = \xi[a_0^{(P)} + \tfrac{1}{6}\pi(a_2^{(P)}/N^2) + \ldots] \tag{2.11}$$

$$a_2^{(P)} = c. \tag{2.12}$$

Relations (2.11) and (2.12) can be combined with (2.9) to determine c. If the central charge c is known, equation (2.11) gives a second determination of ξ, as will be shown in the next section. After we have convinced ourselves that c is indeed equal to unity, we have used (2.11) in order to determine more precisely ξ, assuming $c = 1$.

An interesting part of our study was the numerical determination of the finite-size correction to the ground-state energy in the case of free boundary conditions:

$$-E_0^{(F)}/N = \xi[a_0^{(F)} + a_1^{(F)}/N + \tfrac{1}{6}\pi(a_2^{(F)}/N^2) + \ldots]. \tag{2.13}$$

Here we should obviously have

$$a_0^{(F)} = a_0^{(P)} \tag{2.14}$$

but we have kept two different notations in order to check that our numerical fits give the same result. In equation (2.13), $a_1^{(F)}$ represents the surface energy and $a_2^{(F)}$ is the constant of interest. As will be shown, all the numerical data suggest

$$a_2^{(F)} = \tfrac{1}{4}c \tag{2.15}$$

where c is again the central charge of the Virasoro algebra.

3. Numerical results

3.1. Corrections to the ground-state energy

We have considered the quantum chains defined by (1.1) and have computed the various quantities defined in the previous section using Van den Broeck-Schwartz (1979) approximants. (In all cases we have used chains from two up to ten sites.)

In table 1 we list the approximants for the quantities $\xi a_0^{(P)}$, $2\pi\xi a_2^{(P)}$, $\xi a_0^{(F)}$, $\xi a_1^{(F)}$ and $2\pi\xi a_2^{(F)}$ defined by equations (2.11) and (2.13) (for later convenience we have given the values for $2\pi\xi a_2^{(P)}$ and $2\pi\xi a_2^{(F)}$ instead of $\xi a_2^{(P)}$and $\xi a_2^{(F)}$). In the case of the four-state Potts model ($\varepsilon = 1$) the ground-state energy for the infinite volume case is known exactly. For the Ising case ($\varepsilon = 0$) the leading finite-size corrections are also known exactly. The values of ε have been chosen both outside the critical fan ($\varepsilon \geqslant -\sqrt{2}/2$) and inside the fan ($\varepsilon \leqslant -\sqrt{2}/2$). For $\varepsilon < -\sqrt{2}/2$ we have stayed on the self-dual line $\lambda = 1$ with the exception of the point $\varepsilon = -\frac{39}{41}$ where we have also considered the point with $\lambda = 0.5$. In order to check the errors in our approximants, the calculations have been repeated at the dual point $\varepsilon = -\frac{39}{41}$, $\lambda = 2$ and we have found results fully consistent with those at $\lambda = 0.5$. In table 2 we give the approximants for $P_1^{(P)}$, $P_2^{(P)}$, $P_1^{(F)}$, $P_2^{(F)}$ and $R^{(1)}$.

We now determine the value of the central charge c. Using equations (2.9) and (2.11) we have

$$R^{(1)}/2\pi\xi a_2^{(P)} = 1/c. \tag{3.1}$$

In table 3 we list the values of this ratio and in the whole domain of ε we find it

Table 1. Van den Broeck-Schwartz approximants for $\xi a_0^{(P)}$, $2\pi\xi a_2^{(P)}$ defined by equation (2.11) and for $\xi a_0^{(F)}$, $\xi a_1^{(F)}$ and $2\pi\xi a_2^{(F)}$ defined by equation (2.13) for various values of ε. The quantities with an asterisk are exact.

ε	$\xi a_0^{(P)}$	$\xi a_0^{(F)}$	$2\pi\xi a_2^{(P)}$	$\xi a_1^{(F)}$	$2\pi\xi a_2^{(F)}$
1	$2\ln 2-\frac{1}{2}^*=$	$2\ln 2-\frac{1}{2}^*$	4.9345-	(−0.1888)-	1.03-
	0.886 294		4.9370	(−0.1891)	1.12
$\frac{3}{5}$	0.969 312-	0.963 14-	5.323-	(−0.215)-	1.27-
	0.969 315	0.963 18	5.325	(−0.218)	1.29
$\frac{1}{3}$	1.064 398-	1.064 396-	5.6695-	(−0.259)-	1.40-
	1.064 399	1.064 400	5.6696	(−0.261)	1.42
0	$4/\pi^*$	$4/\pi^*$	$2\pi^*$	$1-2^*/\pi$	$\frac{1}{2}\pi^*$
$-\frac{1}{2}$	2.196 152-	2.196 145-	8.1620-	(−0.8256)-	2.02-
	2.196 153	2.196 155	8.1621	(−0.8257)	2.04
$-\frac{2}{3}$	3.161 74	3.161 750-	9.5928-	(−1.3106)-	2.39-
	3.161 75	3.161 755	9.5930	(−1.3108)	2.40
$-\sqrt{2}/2$	3.566 30-	3.560-	10.10-	(−1.5135)-	2.50-
	3.566 32	3.569	10.12	(−1.5140)	2.53
$-\frac{39}{41}$	20.563 6-	20.564-	23.0-	(−10.00)-	6.0-
$(\lambda=1)$	20.564 1	20.565	26.0	(−10.04)	6.5
$-\frac{39}{41}$	15.428-	15.429-	16.2-	(−4.919)-	4.1-
$(\lambda=0.5)$	15.430	15.431	16.6	(−4.923)	4.4
$-\frac{99}{101}$	50.543-	50.500-	30.5-	(−24.95)-	9.5-
	50.545	50.542	33.0	(−25.05)	10.7
$-\frac{199}{201}$	100.530 6-	100.523-	36.0-	(−49.95)-	9.0-
	100.531 1	100.531	41.0	(−50.10)	14.0

consistent with one. As mentioned in the previous section, this is the expected value for c. A closer inspection of the values obtained for $a_2^{(P)}$ and $a_2^{(F)}$ (see table 1) suggest the relation

$$a_2^{(P)} = 4a_2^{(F)}. \tag{3.2}$$

The validity of equation (3.2) can be seen in table 3. Its apparent failure at $\varepsilon = 1$ can be explained by the known poor convergence which occurs for the four-state Potts model. Relation (3.2) was also checked for the three-state Potts model (von Gehlen and Rittenberg 1986b) where it is obtained with much better precision. In the Ising case the explicit formulae (see, e.g., Burkhardt and Guim 1985) satisfy equation (3.2).

3.2. Scaling dimensions of the order operators (bulk)

Let us now analyse the behaviour of the approximants corresponding to the order operators. In order to facilitate our job, in table 4 we have organised the data in a suggestive way. From the observation that the ratio $8P_1^{(0)}(2\pi\xi a_2^{(P)})^{-1}$ is equal to unity we obtain

$$x_1 = \tfrac{1}{8} \tag{3.3}$$

in agreement with equation (1.3) if we identify x_1 with x_H. We now notice that the ratio $P_2^{(0)}(4P_1^{(0)})^{-1}$ behaves like $\frac{1}{2}x_T$ for $-\sqrt{2}/2 \leqslant \varepsilon \leqslant 1$ and like $2/x_T$ for $-1 < \varepsilon \leqslant -\sqrt{2}/2$. Since for $P_2^{(0)}$ we have determined only the lowest excitation we conclude that we must have here *two* operators with scaling dimensions x_2^{A} and x_2^{B} respectively. These show

Table 2. Van den Broeck-Schwartz approximants for $P_1^{(0)}$ and $P_2^{(0)}$ defined by equation (2.6), $P_1^{(F)}$ and $P_2^{(F)}$ defined by equation (2.7) and for $R^{(1)}$ defined by equation (2.9). The quantities with an asterisk are exact, those with a dagger are approximants with a poor convergence.

ε	$P_1^{(0)}$	$P_2^{(0)}$	$P_1^{(F)}$	$P_2^{(F)}$	$R^{(1)}$
1	0.63-	0.63-	2.2-	2.2-	4.94-
	0.64	0.64	2.3	2.3	4.97
$\frac{3}{5}$	0.664 4-	0.943 77-	1.871-	2.658-	—
	0.665 8	0.943 80	1.876	2.661	
$\frac{1}{3}$	0.708 65-	1.165 25-	1.7232-	2.827-	5.67
	0.708 75	1.165 32	1.7236	2.840	
0	$\frac{1}{4}\pi^*$	$\frac{1}{2}\pi^*$	$\frac{1}{2}\pi^*$	π^*	$2\pi^*$
$-\frac{1}{2}$	1.020 438-	3.060 7-	1.3606-	4.0045-	8.165
	1.020 442	3.060 8	1.3607-	4.6733-	
$-\frac{2}{3}$	1.19909-	4.4789-	1.2841-	4.6726-	9.593
	1.199 15	4.4792	1.2842	4.6733	
$-\sqrt{2}/2$	1.2634-	5.0555-	1.255-	4.95-†	10.105-
	1.2640	5.0565	1.266	5.2	10.115
$-\frac{39}{41}$	2.697-	4.405-	1.08-†	4.36-	21.0-
$(\lambda = 1)$	2.700	4.415	1.12	4.40	22.5
$-\frac{39}{41}$	1.926 15-	3.36-	0.8403-	3.365-	15.7-
$(\lambda = 0.5)$	1.926 17	3.37	0.8411	3.371	15.9
$-\frac{99}{101}$	4.08-	4.263 5-	1.0513-	4.1977-	30.0-†
	4.10	4.265 0	1.0515	4.1982	37.0
$-\frac{199}{201}$	5.6-	4.186-	1.034-	4.147 055-	40.0-†
	5.7	4.190	1.040	4.147 056	52.0

Table 3. The ratio $4a_2^{(F)}/a_2^{(P)}$ and $R^{(1)}/2\pi\xi a_2^{(P)}$ for various values of ε. * and † as in table 2.

ε	1	$\frac{3}{5}$	$\frac{1}{3}$	0	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\sqrt{2}/2$	$-\frac{39}{41}$ $(\lambda = 1)$	$-\frac{39}{41}$ $(\lambda = 0.5)$	$-\frac{99}{101}$	$-\frac{199}{201}$
$4a_2^{(F)}/a_2^{(P)}$	0.88	0.96	0.995	1*	0.99	0.99	0.99	1.0	1.04	1.2	1.2
$R^{(1)}/2\pi\xi a_2^{(P)}$	0.998	—	1.00	1*	1.000	1.000	0.9995	0.9	0.96	1.05†	1.2†

the following behaviour:

$$x_2^A = \tfrac{1}{4}x_T \tag{3.4}$$

$$x_2^B = 1/x_T. \tag{3.5}$$

The fact that for $\varepsilon = -\sqrt{2}/2$ there is a level crossing was explicitly seen. The operator x_2^A can be identified with the operator x_P of equation (1.3). The operator x_2^B is new.

3.3. Scaling dimensions of the order operators (surface)

Let us now turn our attention to the surface exponents. Since $P_1^{(0)}(P_1^{(F)})^{-1}$ is equal to $\frac{1}{2}x_T$ we have

$$x_{1,s} = 1/2x_T. \tag{3.6}$$

Table 4. Ratios of approximants which help to derive equations (3.3)-(3.5). The function $x_T(\varepsilon)$ is defined by equation (1.4).

ε	$\frac{1}{2}x_T$	$P_1^{(0)}/P_1^{(F)}$	$4P_1^{(0)}/P_2^{(F)}$	$P_2^{(0)}/4P_1^{(0)}$	$8P_1^{(0)}/2\pi\xi a_2^{(P)}$	$2/x_T$
1	0.25	0.28	1.2	0.25*	1.0*	—
$\frac{3}{5}$	0.354 694	0.3551	1.0155	0.354 7	0.999	—
$\frac{1}{3}$	0.411 067	0.4112	1.00	0.411 0	1.014	—
0	0.5	0.5*	1.0*	0.5*	1.0*	—
$-\frac{1}{2}$	0.75	0.7499	1.019	0.749 86	0.9999	—
$-\frac{2}{3}$	0.933 810	0.9333	1.026	0.933 82	1.0000	—
$-\sqrt{2}/2$	1.0	1.0188	0.99†	1.000 2	0.99	1.0
$-\frac{39}{41}$ ($\lambda = 1$)	2.504 208	2.7†	2.64	0.408	0.9	0.399
$-\frac{39}{41}$ ($\lambda = 0.5$)	2.504 208	2.29	2.29	0.437	0.94	0.399
$-\frac{99}{101}$	3.940 046	3.92	3.92	0.26	1.04	0.254
$-\frac{199}{201}$	5.562 847	5.5	5.5	0.18	1.1	0.180

The ratio $4P_1^{(0)}(P_2^{(F)})^{-1}$ is equal to unity for $-\sqrt{2}/2 \leq \varepsilon \leq 1$ and to $\frac{1}{2}x_T$ for $-1 < \varepsilon \leq -\sqrt{2}/2$. This means that we again have *two* surface exponents $x_{2,s}^{A}$ and $x_{2,s}^{B}$:

$$x_{2,s}^{A} = 1 \tag{3.7}$$

$$x_{2,s}^{B} = 2/x_T. \tag{3.8}$$

The results described by equations (3.6)-(3.8) are very interesting. Firstly, these confirm the prediction by Cardy (1986b) that for the Kosterlitz-Thouless phase transition ($\varepsilon = -\sqrt{2}/2$) we have $x_{1,s} = \frac{1}{4}$. At the same time for the four-state Potts transition ($\varepsilon = 1$) we get $x_{1,s} = x_{2,s}^{A} = 1$, again in agreement with a prediction by Cardy (1984b). Two previous determinations of $x_{1,s}$ gave $x_{1,s} = 0.78$ (Droz *et al* 1985) and $x_{1,s} = 0.9$ (von Gehlen *et al* 1986) but now the issue is settled.

3.4. Scaling dimensions of the energy density

As the reader might have noticed we have not given values for the approximants corresponding to $R^{(0)}$ (see equation (2.8)) which would determine x_t. For $-\sqrt{2}/2 \leq \varepsilon \leq 1$, the prediction $x_t = x_T$ (see equation (1.3)) was already confirmed by previous calculations. We find that for $-1 < \varepsilon < -\sqrt{2}/2$ the second lowest level in the $Q = 0$ sector belongs to momentum $P \neq 0$. Using the gap in the $Q = 0$, $P = 0$ sector (this uses the third level of $Q = 0$) $x_t = 2.000$ for $\varepsilon = -\sqrt{2}/2$ as expected. In the critical fan region, level crossings cause numerical problems, so we have no reliable results for x_t there.

3.5. Scaling dimensions of the para-fermionic operators

We now consider the para-fermionic operators and make use of equation (2.10). The approximants for $P_1^{(1)}$, $P_1^{(2)}$ and $P_2^{(2)}$ are given in table 5. The scaling dimensions for the operators with spin $\frac{1}{4}$ are

$$x_{1,1} = \frac{P_1^{(1)}}{2\pi\xi a_2^{(P)}}. \tag{3.9}$$

Table 5. Van den Broeck-Schwartz approximants for the quantities $P_1^{(1)}$, $P_1^{(2)}$ and $P_2^{(2)}$ defined by equation (2.10). The ratio $P_1^{(1)}/2\pi\xi a_2^{(P)}$ gives the scaling dimensions $x_{1,1}$ of the spin-$\frac{1}{4}$ para-fermion. The ratio $4P_1^{(2)}/5\pi\xi a_2^{(P)}$ is computed in order to derive equation (3.11). The ratio $P_2^{(2)}/2\pi\xi a_2^{(P)}$ is computed in order to derive equations (3.13) and (3.14). The function $x_T(\varepsilon)$ is defined by equation (1.4).

ε	$P_1^{(1)}$	$P_1^{(2)}$	$P_2^{(2)}$	$P_1^{(1)}/2\pi\xi a_2^{(P)}$	$\frac{4}{5}P_1^{(2)}/\pi\xi a_2^{(P)}$	$P_2^{(2)}/2\pi\xi a_2^{(P)}$	x_T
1	2.45- 2.55	2.97- 3.03	2.22- 2.27	0.51	0.973	0.46	0.5
$\frac{3}{5}$	2.1105- 2.1114	3.32- 3.33	3.77	0.396 6	0.999	0.708	0.7093
$\frac{1}{3}$	2.01- 2.02	3.44- 3.54	4.66	0.355	0.98	0.821	0.8221
0	1.963 3309	3.9267	6.28	0.312 4738	0.999 92	0.999	1
$-\frac{1}{2}$	2.125 54	5.1013	8.52	0.260 41	1.000 0	1.04	1.5
$-\frac{2}{3}$	2.404	5.997	9.61	0.250 6	1.000	1.00	—
$-\sqrt{2}/2$	2.528 2	6.32	—	0.250 1	1.000	—	—
$-\frac{39}{41}$ $(\lambda = 1)$	8.009- 8.011	13.85- 13.95	—	0.33	1.01	—	—
$-\frac{39}{41}$ $(\lambda = 0.5)$	5.43- 5.45	9.7- 9.95	—	0.33	0.96	—	—

Here we have been unable to make an educated guess of the analytic form of the ε dependence of $x_{1,1}$. The value for $\varepsilon = 1$ corresponds to the conjectured value (von Gehlen *et al* 1986, Nienhuis and Knops 1985) $x_{1,1} = \frac{17}{32}$. We notice that for $\varepsilon = -\sqrt{2}/2$, $x_{1,1}$ reaches its obvious lower bound

$$x_{1,1} \geqslant \tfrac{1}{4}. \tag{3.10}$$

The scaling dimension of the spin-$\frac{1}{2}$ operator is simply

$$x_{1,2} = \tfrac{5}{8} \qquad \text{(independent of } \varepsilon\text{)}. \tag{3.11}$$

The value (3.11) for $\varepsilon = 1$ only was known (von Gehlen *et al* 1986, Nienhuis and Knops 1985). The statement (3.11) is confirmed by the fact that the ratio

$$\tfrac{4}{5}P_1^{(2)}/\pi\xi a_2^{(P)} = \tfrac{8}{5}x_{1,2} \tag{3.12}$$

is equal to unity for all ε (see table 5).

A simple inspection of the values of $P_2^{(2)}(2\pi\xi a_2^{(P)})^{-1}$ given in table 5 suggest that we have two operators corresponding to $P_2^{(2)}$ with anomalous dimensions $x_{2,2}^{A}$ with $x_{2,2}^{B}$ satisfying

$$x_{2,2}^{A} = x_T \tag{3.13}$$

$$x_{2,2}^{B} = 1. \tag{3.14}$$

This closes our analysis of the critical exponents.

3.6. The Ising line

Before proceeding with our conclusions we would like to end this section with a technical digression. Using the fact that the Ising line separating the ferromagnetic phase (I in figure 1) and the partially ordered phase (II in figure 1) is not known analytically we ask ourselves the following question: to what precision can the central

charge of the Virasoro algebra ($c = \frac{1}{2}$ in the Ising case) be determined in the case where the critical point is not known exactly? In order to answer our question we have determined λ_{cr} for several values of $\varepsilon(\varepsilon > 1)$ using the standard method of looking at intercepts of the curves $P_1^{(0)}$ as a function of λ for successive values of the length N of the quantum chain. Once the critical point was determined, we have used $R^{(1)}$ (see equation (2.9)) to determine ξ and equation (2.11) to determine $\xi a_2^{(P)}$. Their ratio, according to equation (2.12), determines c.

In table 6 we list the values of λ_{cr}, $\xi a_0^{(P)}$, $\xi a_2^{(P)}$ and ξ for four values of ε. For $\varepsilon = \frac{5}{3}$ the convergence is very poor and we have not given any approximant. This can probably be explained by the fact that the Ising line starts at $\varepsilon = 1$ with zero slope. At $\varepsilon = 3$ however, where λ_{cr} is determined with only four digits, the estimate for c is quite good, $c = 0.502$; it is 0.509 for $\varepsilon = 1.9$ and 0.5004 for $\varepsilon = 39$. From this exercise the reader can get a feeling of what can be expected in other applications where the critical point is not known exactly.

Table 6. The Ising line. Van den Broeck-Schwartz approximants for $\xi a_0^{(P)}$, $\xi a_2^{(P)}$ (see equation (2.11)) and ξ derived from $R^{(1)}$ (see equation (2.9)).

ε	λ_{cr}	$\xi a_0^{(P)}$	$\xi a_2^{(P)}$	ξ
$\frac{5}{3}$	1.049 02(4)	0.852	—	—
3	1.572 (1)	1.097 6-	0.3902-	0.7765-
		1.097 9	0.3906	0.7784
19	9.501 45 (10)	5.118 15-	0.4750-	0.932-
		5.118 25	0.4754	0.936
39	19.500 33 (5)	10.127 1-	0.4874-	0.9742-
		10.127 4	0.4878	0.9748

4. Conclusions

The one-dimensional four-state quantum chain defined by the Hamiltonian (1.1) was shown to be a good laboratory for systems having continuously varying exponents. Because of its connection with the six-vertex model (Lieb 1967a, b) and the Thirring model its properties can be easier understood. The conclusions drawn from the study of this system are certainly useful for the understanding of other systems with six and eight states which apparently have very similar properties (von Gehlen and Rittenberg 1986a). Let us sum up what we have learnt.

(i) Using equation (2.9) which fixes the normalisation of the Hamiltonian and the corrections to the ground-state energy with periodic boundary conditions (equation (2.11)) we have verified that the central charge c of the Virasoro algebra corresponding to the critical line $\lambda = 1$ $(-\sqrt{2}/2 \leqslant \varepsilon \leqslant 1)$ and the critical fan (see figure 1) is equal to one everywhere. For the Ising lines (see figure 1) it was checked that $c = \frac{1}{2}$ as expected. This check is interesting because in this case the critical line is not known analytically and it is suggestive for the errors in the determination of c.

(ii) We have found empirically that the quadratic correction to the ground-state energy per site in the case of free boundary conditions (see equation (2.13)) is also related to the central charge c (see equation (2.14)). The relation (2.14) was also checked for the three-state Potts model and is probably of general validity.

(iii) The scaling dimensions for the order operator related to the charge-one sector (magnetic field operator) are given by equation (3.3) for the bulk and by equation (3.6) for the surface. For the charge-two sector there are two operators both for the bulk (equations (3.4) and (3.5)) and for the surface (equations (3.7) and (3.8)). One of them has smaller scaling dimensions for $\varepsilon > -\sqrt{2}/2$, the other for $\varepsilon < -\sqrt{2}/2$ (inside the critical fan).

(iv) We have determined numerically the scaling dimension of the spin-$\frac{1}{4}$ (charge one) operator (see table 5) but we have not found an analytic expression for it. We notice that the scaling dimensions are $\frac{17}{32}$ for the four-state Potts model ($\varepsilon = 1$) and $\frac{1}{4}$ for the Kosterlitz-Thouless ($\varepsilon = -\sqrt{2}/2$) transition. The spin-$\frac{1}{2}$ (charge one) operator has scaling dimensions $\frac{5}{8}$ for all values of ε. Finally we have determined two (charge two) operators with scaling dimensions given by equations (3.13) and (3.14).

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Note added. After submitting this paper we learnt that Blöte *et al* (1986) have given a proof of the conjecture contained in equation (2.14).

Further numerical studies have shown that the exponent $x_{1,1}$ (see equation (3.9)) has the expression:

$$x_{1,1} = \frac{4 + x_T^2}{16 x_T}.$$

We have also found that the operator with scaling dimension $x_{2,2}^{A}$ (see equation (3.13)) has spin zero while the operator with scaling dimension $x_{2,2}^{B}$ (see equation (3.14)) has spin one.

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